

Spectral Gaps of Spin-orbit Coupled Particles in Deformed Traps

O V Marchukov, A G Volosniev, D V Fedorov, A S Jensen
and N T Zinner

Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C,
Denmark

Abstract. We consider a spin-orbit coupled system of particles in an external trap that is represented by a deformed harmonic oscillator potential. The spin-orbit interaction is a Rashba interaction that does not commute with the trapping potential and requires a full numerical treatment in order to obtain the spectrum. The effect of a Zeeman term is also considered. Our results demonstrate that variable spectral gaps occur as a function of strength of the Rashba interaction and deformation of the harmonic trapping potential. This implies that the many-body physics of spin-orbit coupled systems can be manipulated by variation of the external trapping potentials.

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1. Introduction

The last decade is associated with breakthroughs in ultracold atomic and state-of-the-art optical lattices experiments, which provided not only extremely important data for understanding of the fundamentals of quantum mechanics, but also introduce a sophisticated set of tools for investigation of exceptionally pure and tunable quantum systems [1, 2, 3, 4]. Recently, this toolbox has been expanded to also include the possibility of applying controllable gauge fields to both bosonic [5, 6, 7, 8, 9] and fermionic systems [10, 11] (see references [12] or [13] for short reviews). Spin-orbit coupling is a prime example of a non-abelian potential that plays an important role throughout physics. Examples are the spin-orbit splittings in atomic and nuclear spectra, and the distinct effect it imposes on the band structure of solid-state systems that has lead to great recent advances in the exploration of materials with robust metallic surface states, the so-called topological insulators [14, 15].

In many experiment with ultracold atoms, the external trapping potentials, while present, are often quite shallow and can be ignored for many purposes or one can include them using a local density approximation. However, some recent experiments have demonstrated that tight trapping is not only possible but also a very exciting possibility as this brings the physics one can study close to what is known from atomic or nuclear structure. Some recent theoretical papers have shown that the trapping potential can play an important role in spin-orbit coupled systems [16, 17, 18, 19]. The cited works consider the case where the trapping potential is isotropic in all three dimensions [16] or isotropic in the plane where the spin-orbit coupling acts [17, 18, 19]. In the current presentation we extend this discussion by considering also the case where the trapping potential can be deformed.

Our findings show that deformation of the external potential has a decisive effect on the spectral density and can lead to the opening and (near) closing of the energy gaps in the single-particle spectrum. In the strongly deformed effectively one-dimensional limit the level spacing is completely determined by the shallow trap harmonic frequency at low energy. However, for small deformation one can find regimes where the spectral gaps will almost close and produce a (quasi)-continuum. This implies that deformation can be used as a control parameter for the level density. In the case of a many-body system with non-abelian gauge potentials, the spectral density plays a decisive role in trapped systems when exploring many-body phenomena such as exotic pairing and crossover [20, 21, 22, 23, 25], superfluidity and condensation [26, 27, 28, 29, 30], ferromagnetism [31] or quantum Hall states [32, 33, 34, 35, 36, 37, 38, 39]. Our results represent an initial step in exploring these interesting questions for spin-orbit coupled systems in deformed traps.

2. Formalism

We consider the single-particle problem in a deformed three-dimensional (3D) harmonic trap including a Rashba [40] spin-orbit interaction and an applied magnetic field. Here we use the Rashba form of the spin-orbit interaction but the formalism is completely general and applies to all types of spin-orbit coupling including the case where the Rashba-type and Dresselhaus-type [41] contributions are equal (which amounts to a term like $\sigma_x p_y$ in the notation introduced below) which is used in many current cold atomic gas experiments studying spin-orbit effects. The Hamiltonian for

a particle with mass m experiencing a Rashba and Zeeman term is

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2 + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2 z^2 + \alpha_R(\sigma_x p_y - \sigma_y p_x) - \boldsymbol{\mu} \cdot \mathbf{B}, \quad (1)$$

where σ_x and σ_y are 2×2 Pauli matrices and α_R is the strength of the Rashba spin-orbit coupling. We consider a particle with two internal degrees of freedom, which we label as spin up $|\uparrow\rangle$ and spin down $|\downarrow\rangle$. We choose the effective magnetic field as $\mathbf{B} = (0, 0, B)$ and we can write the Zeeman term as $-\mu B \sigma_z$ with σ_z the remaining Pauli matrix and μ is the effective magnetic moment. For the harmonic trapping potentials we assume different trapping frequencies in all directions, ω_x , ω_y and ω_z . However, since we take the Rashba term to act only in the xy -plane the z -direction effectively decouples from the problem. Note that in the case of $B = 0$, the Hamiltonian is symmetric under time-reversal and for a single particle Kramer's theorem dictates a two-fold degeneracy.

The Hamiltonian can be written in the matrix form

$$\begin{pmatrix} H_{0x} + H_{0y} - \mu B & \alpha_R(p_y + ip_x) \\ \alpha_R(p_y - ip_x) & H_{0x} + H_{0y} + \mu B \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = E \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad (2)$$

where we have introduced the short-hand $H_{0x} = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2$ and similarly for H_{0y} . This constitutes an effective two-dimensional (2D) problem. The oscillator energy in z -direction can be considered as a parameter that can be included in the total energy, E . The natural basis in which to expand our wave-functions are eigenfunctions of the 2D harmonic oscillator. Thus,

$$\psi_\uparrow = \sum_{n_x, n_y} a_i(n_x, n_y) |n_x, n_y, \uparrow\rangle \quad (3)$$

$$\psi_\downarrow = \sum_{n_x, n_y} b_i(n_x, n_y) |n_x, n_y, \downarrow\rangle, \quad (4)$$

where

$$|n_x, n_y\rangle = \sqrt{\frac{\pi}{2^{n_x+n_y} n_x! n_y!}} e^{-\frac{x^2/b_x^2 + y^2/b_y^2}{2}} H_{n_x}\left(\frac{x}{b_x}\right) H_{n_y}\left(\frac{y}{b_y}\right), \quad (5)$$

with $b_x = \frac{\hbar}{m\omega_x}$ and $b_y = \frac{\hbar}{m\omega_y}$. Introducing the standard ladder operators $a_{x,y}$ and $a_{x,y}^\dagger$ that obey

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega_x}}(a_x^\dagger + a_x), & p_x &= \sqrt{\frac{m\hbar\omega_x}{2}}(a_x^\dagger - a_x), \\ y &= \sqrt{\frac{\hbar}{2m\omega_y}}(a_y^\dagger + a_y), & p_y &= \sqrt{\frac{m\hbar\omega_y}{2}}(a_y^\dagger - a_y), \end{aligned} \quad (6)$$

the Hamiltonian now reads

$$\begin{pmatrix} \gamma(\hat{N}_x + \frac{1}{2}) + (\hat{N}_y + \frac{1}{2}) - \frac{\mu B}{\hbar\omega_y} & \beta(i\sqrt{\omega_y}(a_y^\dagger - a_y) - \sqrt{\gamma}(a_x^\dagger - a_x)) \\ \beta(i(a_y^\dagger - a_y) + \sqrt{\gamma}(a_x^\dagger - a_x)) & \gamma(\hat{N}_x + \frac{1}{2}) + (\hat{N}_x + \frac{1}{2}) + \frac{\mu B}{\hbar\omega_y} \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \epsilon \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad (7)$$

where we have introduced the notation $\hat{N}_x = a_x^\dagger a_x$ and $\hat{N}_y = a_y^\dagger a_y$, the dimensionless Rashba coupling $\beta = \alpha_R \sqrt{\frac{m}{2\hbar\omega_y}}$, and $\epsilon = \frac{E}{\hbar\omega_y}$. The ratio of the harmonic oscillator

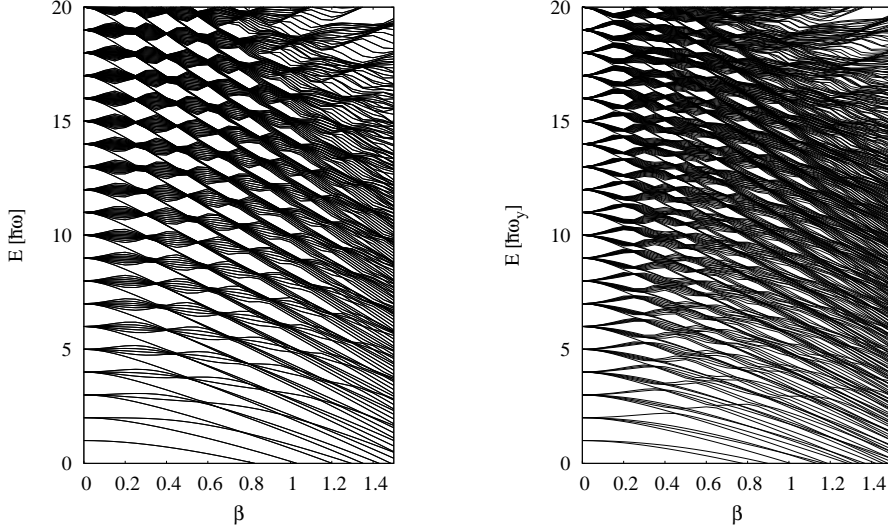


Figure 1. Energy as function of the dimensionless spin-orbit coupling parameter β for the case of equal frequencies $\omega = \omega_x = \omega_y$ with no Zeeman shift (left panel) and including a Zeeman shift of magnitude $\mu B = \hbar\omega$.

frequencies is $\gamma = \frac{\omega_x}{\omega_y}$. We will be using $\hbar\omega_y$ as the unit of energy. The linear system of equations for the coefficients a and b becomes

$$\begin{aligned}
 & \left(\gamma \left(n_x + \frac{1}{2} \right) + n_y + \frac{1}{2} \right) - \epsilon) a(n_x, n_y) + \beta \left[i\sqrt{n_y} b(n_x, n_y - 1) - i\sqrt{n_y + 1} b(n_x, n_y + 1) \right. \\
 & \quad \left. - \sqrt{\gamma} \sqrt{n_x} b(n_x - 1, n_y) + \sqrt{\gamma} \sqrt{n_x + 1} b(n_x + 1, n_y) \right] = 0 \\
 & \left(\gamma \left(n_x + \frac{1}{2} \right) + n_y + \frac{1}{2} \right) - \epsilon) b(n_x, n_y) + \beta \left[i\sqrt{n_y} a(n_x, n_y - 1) - i\sqrt{n_y + 1} a(n_x, n_y + 1) \right. \\
 & \quad \left. + \sqrt{\gamma} \sqrt{n_x} a(n_x - 1, n_y) - \sqrt{\gamma} \sqrt{n_x + 1} a(n_x + 1, n_y) \right] = 0,
 \end{aligned} \tag{8}$$

where $\epsilon_o(n_x, n_y) = \gamma \left(n_x + \frac{1}{2} \right) + \left(n_y + \frac{1}{2} \right)$. This set of equations cannot be solved analytically and one has to resort to numerical methods. In the symmetric case where $\gamma = 1$, one could also have used a basis based on the solution of the harmonic oscillator potential in cylindrical coordinates [17, 18, 19]. However, this is not appropriate in our case since the cylindrical symmetry is broken in the plane for $\gamma \neq 1$ and the problem is therefore better handled using the Cartesian basis expansion presented above.

3. Results

We now present the results of our study of the single-particle spectral structure in the presence of a Rashba spin-orbit term and with an external trap that can be

deformed. All the results presented here are in the regime $0 \leq \beta \leq 1.5$. It is a simple matter to go to even higher Rashba couplings but it requires the use of a bigger single-particle basis in order for all states to properly converge. The results we present below have all been obtained by using a basis with about 700 single-particle states. However, due to deformation the maximum number of quanta, $\max(n_x)$ and $\max(n_y)$, is generally not equal to each other since we cut the basis in energy space, i.e. $\max(\epsilon_0(n_x, n_y)) = \gamma(\max(n_x) + 1/2) + (\max(n_y) + 1/2)$ is our cut-off that restricts the values of n_x and n_y .

3.1. Symmetric Trapping Potential

In the left panel in figure 1 we show the single-particle energy spectrum in a two-dimensional spherical harmonic oscillator as function of the coupling strength, β . The oscillator degeneracies that are well-known for $\beta = 0$ are lifted as β increases and the oscillator shells become more and more mixed. Each level is still doubly degenerate due to time reversal symmetry mentioned above. The lowest levels decrease and for sufficiently large strengths the behavior approaches a parabolic dependence on β that can be found in a semi-classical treatment [16]. The many levels are spread out over energies and overall they cover rather densely the energy space. In many regions the spectrum resembles a continuous distribution where the individual levels cannot be distinguished. The deviations from the regular picture seen in the top right-hand corner on both panels in figure 1 is due to the numerics and the use of a finite basis set. We have numerically checked that this can be remedied by systematic expansion of the basis size.

The key features are found in the low-density pockets of the energy spectrum where no levels are found. We see that we have a spectrum that can open and close 'super' gaps as a function of the Rashba coupling strength. These regions are significant since we would expect that a system with this level density would tend to be more stable with respect to the application of weak interactions between the particles or perturbations from the background since the larger gaps hinder strong mixing of states. In the opposite case where the level density is high we expect to see strong mixing and small perturbations could have large effects. What is particularly remarkable here is the fact that the Rashba term can almost cancel the effects of the trap, i.e. the spacing of trap levels at $\beta = 0$ of one $\hbar\omega$ is strongly reduced at $\beta \sim 0.2$ for total energies above $\sim 5\hbar\omega$. Indeed these sorts of changes are reflected on the properties of a many-body system in a symmetric trap as discussed for the case of condensates in references [17], [18] and [19].

On the right panel in figure 1 we show again the case of equal frequencies, $\omega = \omega_x = \omega_y$, but this time with the inclusion of a Zeeman field of magnitude $\mu B = \hbar\omega$. The Zeeman field will not influence the oscillator levels but only displace the spin components. Note that the absolute ground state of the spectrum now starts at zero energy and then decreases. We have opted to keep the same vertical scale on both panels in figure 1 in order to make a comparison of the overall spectral density at higher energies and therefore the Zeeman shifted ground state level cannot be seen. What we find is that the Zeeman split will tend to counteract the effect of the Rashba term at small β so that we now find a spectrum with large gaps at $\beta \sim 0.2$. However, since the spin splitting sends states up and down in the spectrum it tends to generate a sort of 'two-gap' structure around $\beta \sim 0.2$ with one large and one smaller gap, intertwined by a number of densely spaced states. The conclusion is that both

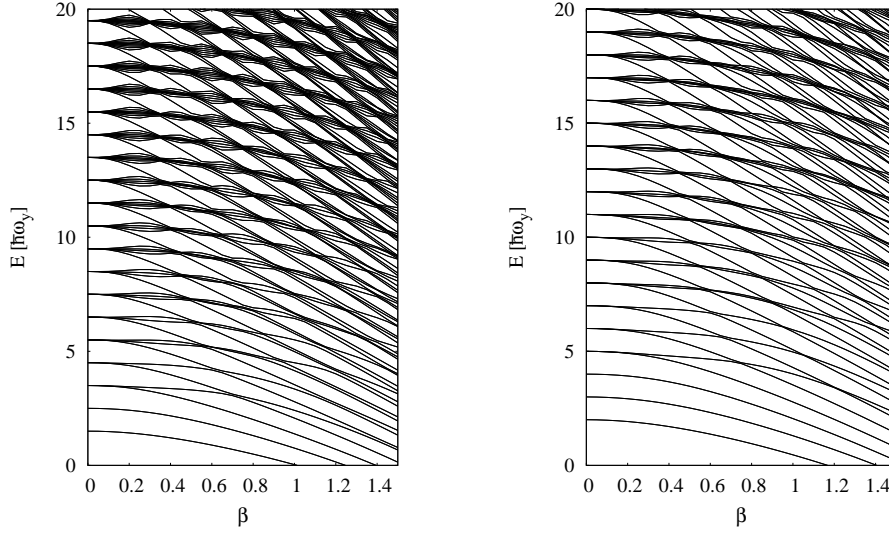


Figure 2. Energy as function of dimensionless spin-orbit coupling parameter β for the case where the oscillator potential is deformed. The left panel has $\gamma = \frac{\omega_x}{\omega_y} = 2$ and the right panel has $\gamma = 3$.

Rashba and Zeeman can be used as an experimental handle on the density of states in the single-particle spectrum as the density and structure can be changed by varying one while keeping the other fixed.

3.2. Deformed Trapping Potentials

We now consider the case of an oscillator potential that is non-spherical corresponding to different frequencies in the x and y directions. In this situation, high degeneracy is found for special frequency ratios of deformed harmonic oscillators. These special configurations occur when the frequency ratios equal ratios of small integers as $2/1$, $3/2$, etc. This is due to the harmonic oscillator spectrum being linear in both the frequencies and quantum numbers in the different spatial directions. The harmonic oscillator has especially high degeneracy compared to deformations of (almost) all other radial shapes. This means that gaps are more likely to occur in the spectrum. The implication for pure oscillators with no Rashba or Zeeman terms is that degeneracies are periodically occurring as function of the frequency ratio, and that degeneracies are washed out in between the highly degenerate points. In figure 2 we show the spectrum for frequency ratios $\gamma = 2$ (left panel) and $\gamma = 3$ (right panel) as a function of Rashba coupling strength, β , with no Zeeman field. For these ratios we clearly see that the spectrum is going towards a more degenerate form with stronger gaps for increasing γ in comparison to the $\gamma = 1$ case shown in figure 1. Indeed for $\gamma = 3$ we have to go up to energies of $15\hbar\omega_y$ and above before we can see the same behavior as for lower energies in the left panel of figure 1. When including a Zeeman

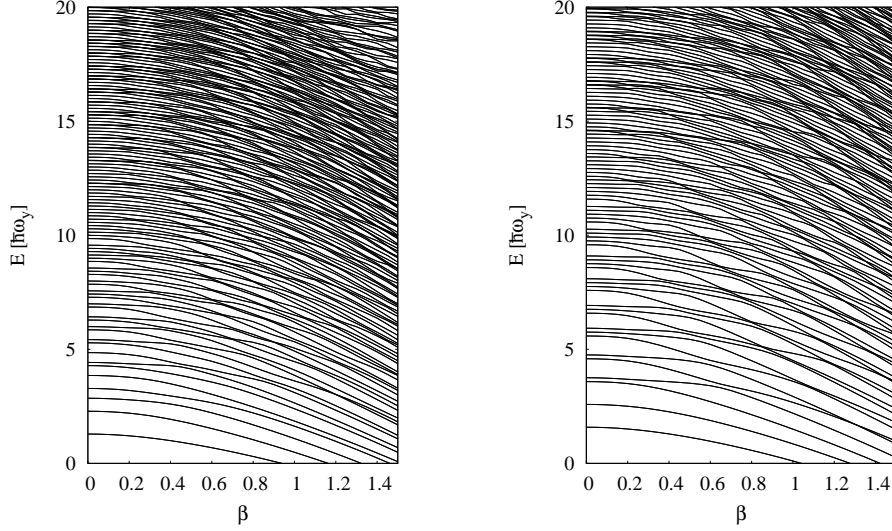


Figure 3. Same as figure 2 but with ratios $\gamma = 1.57$ in the left panel and $\gamma = 2.17$ in the right panel.

term for these ratios (not shown here) we see the same trends as seen from left to right in figure 1, i.e. the spectrum becomes slightly more dense overall and there is competition between Rashba and Zeeman terms.

A completely different scenario is displayed in the case where γ is not a ratio of small integers. In figure 3 we show the single-particle spectrum for $\gamma = 1.57$ (left panel) and $\gamma = 2.17$ (right panel). For these ratios the states are more evenly distributed and for $\gamma = 1.57$ we see an almost constant spectral density for energies of $5\hbar\omega_y$ and above, while for $\gamma = 2.17$ this is not seen until about $10\hbar\omega_y$ and above. Comparing the left and right-hand panels in figure 3, we see an overall tendency for the larger γ to have a smaller overall density of levels since this is closer to the one-dimensional limit that we will return to momentarily. The overall quadratic decrease with β is still seen as in figures 1 and 2. Comparing the results presented in figure 2 for the ratios $\gamma = 2$ and $\gamma = 3$ to those of figure 3 with $\gamma = 1.57$ and $\gamma = 2.17$ we thus conclude that the deformation of the trap is another experimental way to change the spectral structure and density. However, for the latter ratios the influence of the Rashba coupling is diminished somewhat as the density changes in a much more smooth manner as compared to figures 1 and 2. This implies that the choice of deformation is very important when studying the effects of spin-orbit coupling on trapped systems.

3.3. One-dimensional Limit

Increasing the frequency ratio towards infinity separates the Hamiltonian in a low-energy one-dimensional part very weakly coupled to the high-energy one-dimensional

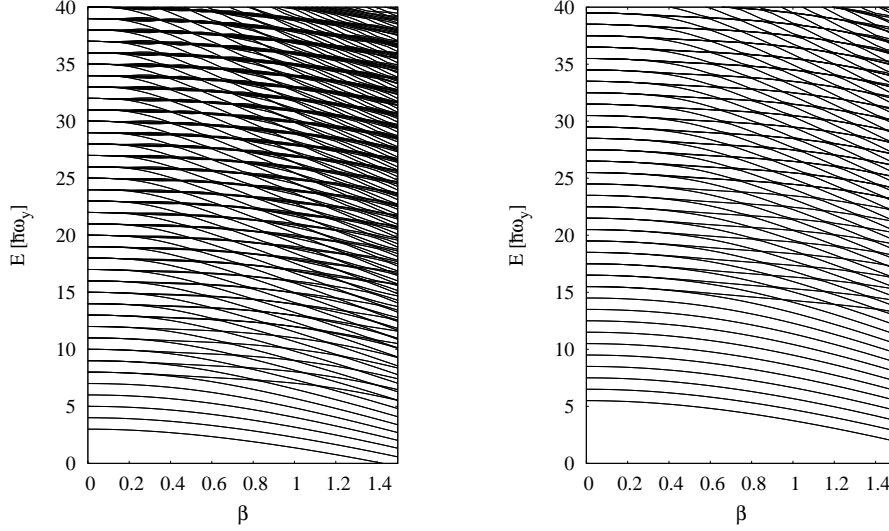


Figure 4. Same as figure 2 and 3 for $\gamma = 5$ (left panel) and $\gamma = 10$ (right panel). These results approach the limit of an effectively one-dimensional system, i.e. $\gamma \gg 1$.

part in the other direction. With our conventions, the limit $\gamma \ll 1$ corresponds to a strongly confined motion along the x -direction and a shallow confinement along the y -direction. These decoupled one-dimensional equations can be solved analytically with the corresponding Rashba couplings. The Pauli matrix for either σ_x or σ_y implies that one should use a basis of eigenfunctions for the operator with the small frequency, i.e. ω_y , since the other direction has a very large oscillator frequency that effectively freezes the motion. This linear combination of equal amplitude spin-up and spin-down spinors immediately decouple the equations of motion into separate oscillator eigenvalue problems. The eigenvectors of σ_x are

$$\Phi_{\pm} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad (9)$$

and these can now be used as the two-component basis instead of the spin-up and spin-down spinors. The one-dimensional Schrödinger equation then becomes

$$\left(\frac{p_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2 \pm \alpha_R p_y - E \right) \Phi_{\pm} f_{\pm}(y) = 0, \quad (10)$$

where $f_{\pm}(y)$ is the radial wave function. We rewrite these equations as

$$\left[\frac{1}{2m}(p_y \pm m\alpha_R)^2 + \frac{1}{2}m\omega_y^2 y^2 - \frac{m\alpha_R^2}{2} - E \right] \Phi_{\pm} f_{\pm}(y) = 0, \quad (11)$$

from which the harmonic oscillator solutions can be directly inferred to be

$$E_{n_y} = -\frac{1}{2}m\alpha_R^2 + \hbar\omega_y(n_y + \frac{1}{2}) \quad (12)$$

The first term is simply the quadratic decrease of the energy with β that we already noted above and that can be obtained by semi-classical means [16]. We can compare this dispersion to the one-dimensional solution with no external trapping potential in the y -direction which is simply

$$E_{k_y} = \frac{\hbar^2 k_y^2}{2m} \pm \alpha_R \hbar k_y, \quad (13)$$

where k_y is the one-dimensional momentum along the y -direction. These two dispersion relations look very different but can be reconciled by recalling that the virial theorem tells us that $\langle \tilde{p}_y^2 \rangle \propto m \hbar \omega_y n_y$ where $\tilde{p}_y = p_y \pm m \alpha_R$ and we thus recover a momentum-dependence of the energy with quadratic and linear terms.

In figure 4 we show the spectral structure as one approaches the one-dimensional limit. In the left panel of figure 4 we have $\gamma = 5$ and in the right panel $\gamma = 10$. The low-energy eigenvalues reduce to the equidistant harmonic oscillator spectrum with the same frequency but shifted quadratically with the spin-orbit coupling strength. The remarkably simple emerging feature is that the lowest part of the spectrum for an even modest deformation already resembles the equidistant one-dimensional limit. This can be clearly seen by making a comparison of figure 4 to figures 2 and 3. As the deformation increases an increasing part of the low-energy spectrum approaches the one-dimensional limit. This feature can be understood from the weak coupling of two oscillators with very different frequencies. The perturbation on the lowest energy states in the spectrum from the lowest of the large frequency (ω_x) states is proportional to the square of the coupling strength divided by the energy difference by second order perturbation theory. This tells us that the one-dimensional limit is approached in the bottom of the spectrum with increasing deformation. This low-energy spectrum is much simpler and much less dense than that of the two-dimension spherical oscillator. Note that the spectrum is still two-fold degenerate due to the time-reversal symmetry, but now it is practically back to the standard equidistant scheme typical of harmonic confinement.

4. Discussion

We have shown that the single-particle spectrum for particles that are subject to a Rashba spin-orbit coupling in a deformed harmonic trap is very dependent on the parameters and that spectral gaps can be tuned by varying the Rashba coupling strength, a perpendicular Zeeman field and by changing the deformation of the trapping potential in the plane where the Rashba coupling acts. In particular, we find that the effect of deformation can be very different depending on whether the ratio of the trapping frequencies in the plane is equal to ratios of small integers or if it is closer to an irrational number. This can change the spectrum from having an intricate structure with several characteristic gap sizes into a more evenly spaced distribution that approximates a continuum.

In this study we have not taken interactions into account. However, it is certainly possible to infer some potential effects of our results on a system with more particles and with interactions. The simple case of a non-interacting Fermi gas becomes very intriguing with the spectra presented here since a Fermi energy that is located in one of the pockets would be more stable than if it occurs in a region of higher state density. With the possibility of tuning this by external means we can explore this physics in a very direct manner. If we consider a weak interaction in a mean-field picture like the

BCS approach, the Rashba coupled case is extremely interesting as has been discussed recently [20, 21, 22, 23, 24, 25]. From BCS theory in finite systems and/or in a trap we know that one needs a critical coupling strength to obtain a superfluid state (in free space the critical value goes to zero). This critical value depends intimately on the density of states and the tunability demonstrated here shows that one can probe exotic superfluid states by using the deformation.

For neutral single-species ultracold atoms the interactions are of van der Waals type and for the low experimental densities this can be considered of essentially zero-range. For a system of fermions with two internal hyperfine states this implies that one only has interaction in the spin singlet channel. The results presented here can be generalized to the two-particle case and one may then ask for the probability to find two fermions in the singlet state as the Rashba coupling strength, the Zeeman field and the deformation is varied. This will then tell us how the interactions are suppressed by the trapping potential in the presence of a spin-orbit coupling. In fact this is the first step toward the study of few-body physics in trapped system with non-abelian gauge potentials, and this is a direction that we will explore in the near future. Some open questions are the spectrum of two and three fermions in a trap with short-range interactions and its dependence on the deformation of the external confinement. Recent work has explored the case of few-boson problems in gauge fields in order to study strongly correlated dynamics akin to quantum Hall systems [42, 43]. We expect that fermions in different trapping geometries should also give rise to interesting strongly correlated states when subjected to external gauge potentials.

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References

- [1] Ketterle W and Zwierlein M W 2008 *Nuovo Cimento Rivista Serie* **31** 247
- [2] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80** 885
- [3] Esslinger T 2010 *Ann. Rev. Cond. Mat. Phys.* **1** 129
- [4] Cirac J I and Zoller P 2012 *Nature Phys.* **8** 264
- [5] Lin Y-J *et al* 2009 *Phys. Rev. Lett.* **102** 130401
- [6] Lin Y-J, Compton R L, Jimenez-Garcia K, Porto J V and Spielman I B 2009 *Nature* **462** 628
- [7] Lin Y-J, Jimenez-Garcia K and Spielman I B 2009 *Nature* **471** 83
- [8] Aidelsburger A 2011 *Phys. Rev. Lett.* **107** 255301
- [9] Zhang J-Y *et al* 2012 *Phys. Rev. Lett.* **109** 115301
- [10] Wang P *et al.* 2012 *Phys. Rev. Lett.* **109** 095301
- [11] Cheuk L *et al.* 2012 *Phys. Rev. Lett.* **109** 095302
- [12] Dalibard J, Gerbier F, Juzeliunas G and Öhberg P 2011 *Rev. Mod. Phys.* **83** 1523
- [13] Zhai H 2012 *Int. J. Mod. Phys. B* **26** 1230001
- [14] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
- [15] Qi X-L and Zhang S-C 2011 *Rev. Mod. Phys.* **83** 1057
- [16] Ghosh S K, Vyasankere J P and Shenoy V B 2011 *Phys. Rev. A* **84** 053629
- [17] Sinha S, Nath R and Santos L 2011 *Phys. Rev. Lett.* **107** 270401
- [18] Hu H, Ramachandran B, Pu H and Liu X-J 2012 *Phys. Rev. Lett.* **108** 010402
- [19] Hu H and Liu X-J 2012 *Phys. Rev. A* **85** 013619
- [20] Vyasankere J P and Shenoy V B 2011 *Phys. Rev. B* **83** 094515
- [21] Yu Z-Q and Zhai H 2011 *Phys. Rev. Lett.* **107** 195305
- [22] Gong M, Tewari S and Zhang C 2011 *Phys. Rev. Lett.* **107** 195303
- [23] Iskin M and Subasi A L 2011 *Phys. Rev. Lett.* **107** 050402
- [24] Han L and Sá de Melo C A R 2012 *Phys. Rev. A* **85** 011606(R)
- [25] Vyasankere J P and Shenoy V B 2012 *New J. Phys.* **14** 043041

- [26] Stanescu T D, Anderson B and Galitski V 2008 *Phys. Rev. A* **78** 023616
- [27] Wang C, Gao C, Jian C-M and Zhai H 2010 *Phys. Rev. Lett.* **105** 160403
- [28] Ho T-L and Zhang S 2011 *Phys. Rev. Lett.* **107** 150403
- [29] Struck J *et al* 2012 *Phys. Rev. Lett.* **108** 225304
- [30] Zhou K and Zhang Z 2012 *Phys. Rev. Lett.* **108** 025301
- [31] Baur S K and Cooper N R 2012 *Preprint* arXiv:1208.6540
- [32] Goldman N and Gaspard P 2007 *Europhys. Lett.* **78** 60001
- [33] Goldman N *et al* 2009 *Phys. Rev. Lett.* **103** 035301
- [34] Anderson B, Stanescu T D and Galitski V 2010 *Phys. Rev. B* **81** 121304(R)
- [35] Cooper N R 2011 *Phys. Rev. Lett.* **106** 175301
- [36] Grass T, Saha K, Sengupta K and Lewenstein M 2011 *Phys. Rev. A* **84** 053632
- [37] Goldman N, Beugeling W and Morais Smith C 2012 *Europhys. Lett.* **97** 23003
- [38] Hauke P *et al* 2012 *Phys. Rev. Lett.* **109** 145301
- [39] Beugeling W, Goldman N and Morais Smith C 2012 *Phys. Rev. B* **86** 075118
- [40] Rashba E I 1960 *Sov. Phys. Solid State* **2** 1109
- [41] Dresselhaus G 1955 *Phys. Rev.* **100** 580
- [42] Juliá-Díaz B *et al* 2011 *Phys. Rev. A* **84** 053605
- [43] Juliá-Díaz B, Grass T, Barberán N and Lewenstein M 2012 *New J. Phys.* **14** 055003